



A note on the second order geometric Rellich inequality on half-space

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Abstract

We prove a two-parameter family of second order geometric Rellich type equalities on the half-space $\mathbb{R}_+^n$. We then use it to derive several refined geometric Rellich type inequalities. We also prove an improved version of the Rellich–Sobolev–Maz’ya type inequality.

Keywords Rellich inequality · Hardy inequality · Factorization of differential operators

Mathematics Subject Classification 26D10 · 46E35 · 35A23

1 Introduction

The main topic of this note is geometric Rellich type inequalities, that is Rellich type inequalities in which the weights are given in terms of the distance to the boundary of the domain. We are particularly concerned with the second order case for which the

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domain is the half-space $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$ so that the distance from $x \in \mathbb{R}_+^n$ to the boundary of the domain is x_n .

Rellich type inequalities are higher-order versions of the Hardy type inequalities. Both Hardy type inequalities and Rellich type inequalities have received a tremendous amount of attention in the literature because of their important roles in many areas of mathematics such as self-adjointness and spectral theory problems of Schrödinger operators with strongly singular potentials, boundary decay and spectral approximation, uncertainty principles, to name just a few. There is a vast literature, especially, in the context of Hardy type inequalities. Hence it is impossible to provide a complete list of references on the subject in our short note. Therefore, we refer the interested reader, for instance, to [11,26,35] for historical backgrounds and to the monographs [3,24,27,28,36,39] for the applications as well as improvements and variants of Hardy type inequalities and Rellich type inequalities.

At the 1954 ICM in Amsterdam [41], Rellich presented the following inequality as the second order version of the well-known Hardy inequality:

$$\int_{\mathbb{R}^n} |\Delta u|^2 dx \geq \left(\frac{n(n-4)}{4} \right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^4} dx. \quad (1)$$

The constant $\left(\frac{n(n-4)}{4} \right)^2$ is sharp but can not be attained in (1) by nontrivial functions. This fact has been investigated by many researchers. Also, much effort has been devoted to the study of improved Hardy–Rellich type inequalities. See [1,2,6–10,13–16,18,19,22,25,29,30,34,37,38,42,44,46,47], among others.

Our main motivation of this article is the results and approaches in [20] where Gesztesy and Littlejohn applied the method of factorization of singular, even-order partial differential operators to prove the following two-parameter family of Rellich type inequalities:

Theorem A. *Let $\alpha, \beta \in \mathbb{R}$ and $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Then*

$$\begin{aligned} \int_{\mathbb{R}^n} |\Delta u|^2 dx &\geq [(n-4)\alpha - 2\beta] \int_{\mathbb{R}^n} \frac{|\nabla u(x)|^2}{|x|^2} dx \\ &\quad - \alpha(\alpha-4) \int_{\mathbb{R}^n} \frac{\left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2}{|x|^2} dx \\ &\quad + \beta[(n-4)(\alpha-2) - \beta] \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^4} dx. \end{aligned}$$

By choosing particular values for α and β , one can recover from Theorem A many known Rellich type inequalities in the literature, including Rellich type inequalities (1) and the Schmincke one-parameter family of Rellich type inequalities in [45].

We note here that factorization of singular partial differential operators has been applied in [23] to give a simple approach to the classical Hardy inequality and in

[21] to the radial Hardy inequality with logarithmic refinements. Recently, in [43], the method of factorization was used to obtain a Hardy, a Hardy–Rellich, and a refined Hardy inequality on general stratified groups and a weighted Hardy inequality on general homogeneous groups. More recently, several Hardy type identities on half-spaces were established in [31,32] using factorizations. For a thorough review, the history and properties of the factorization method, we refer the interested reader to [20].

Rellich type inequalities have also been studied when the distance to the origin $|x|$ is replaced by the distance to the boundary. In particular, on half-space $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$, the following versions of Rellich type inequalities have been studied:

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|\nabla u(x)|^2}{x_n^2} dx \geq \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx. \quad (2)$$

See, for instance, [4,5,40]. Recently, in [17], the following Rellich type equalities have been established to provide a direct understanding to (2):

Theorem B. For $u \in C_0^\infty(\mathbb{R}_+^n)$, one has

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx = \sum_{j=1}^n \int_{\mathbb{R}_+^n} \left| \nabla \left(\frac{1}{\sqrt{x_n}} \frac{\partial u}{\partial x_j} \right) \right|^2 x_n dx$$

and

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx = \frac{1}{4} \int_{\mathbb{R}_+^n} \left| \nabla \left(\frac{u}{\sqrt{x_n^3}} \right) \right|^2 x_n dx + \sum_{j=1}^n \int_{\mathbb{R}_+^n} \left| \nabla \left(\frac{1}{\sqrt{x_n}} \frac{\partial u}{\partial x_j} \right) \right|^2 x_n dx.$$

Theorem C. For $u \in C_0^\infty(\mathbb{R}_+^n)$, one has

$$\int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx = \int_{\mathbb{R}_+^n} \left| \frac{\partial}{\partial x_n} \left(\frac{1}{\sqrt{x_n}} \frac{\partial u}{\partial x_n} \right) \right|^2 x_n dx$$

and

$$\int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx = \frac{1}{4} \int_{\mathbb{R}_+^n} \left| \frac{\partial}{\partial x_n} \left(\frac{u}{\sqrt{x_n^3}} \right) \right|^2 x_n dx + \int_{\mathbb{R}_+^n} \left| \frac{\partial}{\partial x_n} \left(\frac{1}{\sqrt{x_n}} \frac{\partial u}{\partial x_n} \right) \right|^2 x_n dx.$$

It is worthy to note that Theorem C provided exact remainders to the following Rellich type inequalities that can be found, for instance, in [12, Lemma 5.3.1]:

$$\int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx \geq \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx. \quad (3)$$

The principal goal of this paper is to apply the factorization technique to study the refined versions of (2) and (3). Actually, we will set up the following general two-parameter families of the second order geometric Rellich inequalities:

Theorem 1 Let $\alpha, \beta \in \mathbb{R}$. For $u \in C_0^\infty(\mathbb{R}_+^n)$, one has

$$\begin{aligned} & \int_{\mathbb{R}_+^n} |\Delta u|^2 dx + (\alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ & + (\alpha^2 - 2\alpha) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx \\ & = \int_{\mathbb{R}_+^n} \left| \Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right|^2 dx \end{aligned} \quad (4)$$

and

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx + (\alpha^2 - \alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ & = \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right|^2 dx. \end{aligned}$$

By choosing $\alpha = 2$ and $\beta = \frac{9}{4}$ in Theorem 1, we obtain the following improved second order geometric Rellich type inequalities:

Theorem 2 For $u \in C_0^\infty(\mathbb{R}_+^n)$, one has

$$\begin{aligned} & \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{5}{2} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ & = \int_{\mathbb{R}_+^n} \left| \Delta u - \frac{2}{x_n} \frac{\partial u}{\partial x_n} + \frac{9}{4} \frac{u}{x_n^2} \right|^2 dx. \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx - \frac{5}{2} \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ & = \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} - \frac{2}{x_n} \frac{\partial u}{\partial x_n} + \frac{9}{4} \frac{u}{x_n^2} \right|^2 dx. \end{aligned}$$

Obviously, our Theorem 2 implies

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx \geq \frac{5}{2} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx - \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx$$

and

$$\int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx \geq \frac{5}{2} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx - \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx.$$

Noting that from Hardy type inequalities

$$\int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx \geq \frac{9}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx$$

and

$$\int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx \geq \frac{9}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx,$$

we deduce

$$\frac{5}{2} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx - \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx$$

and

$$\frac{5}{2} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx - \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx.$$

Hence, combining the above results, we get the following series of Rellich type inequalities:

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx \geq \frac{5}{2} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx - \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx \geq \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \quad (5)$$

$$\int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx \geq \frac{5}{2} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx - \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx \geq \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx. \quad (6)$$

Let us now explain why we are interested in the values $\alpha = 2$ and $\beta = \frac{9}{4}$ of Theorem 1. Assume that $\alpha^2 - 2\alpha \leq 0$. In this situation, we deduce from (4) that

$$\begin{aligned} & \int_{\mathbb{R}_+^n} |\Delta u|^2 dx + (\alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ & \geq \int_{\mathbb{R}_+^n} \left| \Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right|^2 dx \\ & \geq 0. \end{aligned}$$

Since we are interested in the enhanced form of (2), we will aim for the following inequalities

$$\begin{aligned} & \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx \\ & \geq \int_{\mathbb{R}_+^n} |\Delta u|^2 dx + (\alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ & \geq 0. \end{aligned}$$

Equivalently,

$$\left(2\beta - \alpha - \frac{1}{4} \right) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx \geq (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx. \quad (7)$$

Recall that (see, for instance, [31]):

$$\int_{\mathbb{R}_+^n} \frac{|\nabla u(x)|^2}{x_n^2} dx \geq \frac{9}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx$$

with sharp constant $\frac{9}{4}$. Hence (7) is optimal when $2\beta - \alpha - \frac{1}{4} > 0$ and $(6\beta - 3\alpha\beta + \beta^2) = \frac{9}{4} (2\beta - \alpha - \frac{1}{4})$. Note that the latter is equivalent to

$$\left(\beta + \frac{3}{4} - \frac{3}{2}\alpha \right)^2 = \frac{9}{4} (\alpha^2 - 2\alpha) \leq 0.$$

Hence, in this case, $\alpha = 2$ and $\beta = \frac{9}{4}$ are optimal choices.

Similarly, we are interested in the case

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx \\ & \geq \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx + (\alpha^2 - \alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ & \geq 0. \end{aligned}$$

That is

$$\left(\alpha + 2\beta - \alpha^2 - \frac{1}{4} \right) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx \geq (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx.$$

We note that (see, for instance, [31]):

$$\int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx \geq \frac{9}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx$$

where $\frac{9}{4}$ is the optimal constant. Hence, we deduce

$$\begin{cases} \alpha + 2\beta - \alpha^2 - \frac{1}{4} > 0 \\ \frac{9}{4} (\alpha + 2\beta - \alpha^2 - \frac{1}{4}) \geq (6\beta - 3\alpha\beta + \beta^2) \end{cases}.$$

The second inequality can be rewritten as

$$\left(\beta - \frac{3}{2}\alpha - \frac{3}{4} \right)^2 \leq 0.$$

This yields $\beta = \frac{3}{2}\alpha - \frac{3}{4}$. Then the first inequality becomes $\frac{9}{4} > (\alpha - 2)^2$. Hence $\frac{1}{2} < \alpha < \frac{7}{2}$. In this case, we obtain

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx + \left(\alpha^2 - 4\alpha + \frac{3}{2} \right) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + \left(-\frac{9}{4}\alpha^2 + 9\alpha - \frac{63}{16} \right) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ & = \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \left(\frac{3}{2}\alpha - \frac{3}{4} \right) \frac{u}{x_n^2} \right|^2 dx. \end{aligned}$$

Optimizing $\alpha^2 - 4\alpha + \frac{3}{2}$ yields $\alpha = 2$ and $\beta = \frac{9}{4}$.

Also, since the above argument of choosing α and β seems optimal, we conjecture that $\frac{5}{2}$ in (5) and (6) is sharp, that is, there is no $k > \frac{5}{2}$ and l such that for all $u \in C_0^\infty(\mathbb{R}_+^n)$:

$$\begin{aligned} \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx &\geq \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - k \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + l \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq 0. \\ \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx &\geq \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx - k \int_{\mathbb{R}_+^n} \left| \frac{\partial u}{\partial x_n} \right|^2 \frac{1}{x_n^2} dx + l \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq 0. \end{aligned}$$

Our next purpose is to study an improved version of the Rellich–Sobolev–Maz’ya type inequality. We note that in [3, Chapter 6], the following Rellich–Sobolev–Maz’ya type inequality has been verified:

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq C \left(\int_{\mathbb{R}_+^n} |u|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}}. \quad (8)$$

Nothing is known about the optimal constant of (8) except when $n = 5$. Indeed, in this case, it was proved in [33] that the sharp constant of (8) coincides with the best second order Sobolev constant $S_{5,2}$ in $\mathbb{R}^5$.

Our second main result can be read as follows:

Theorem 3 Assume that $n \geq 5$ and $\varepsilon > 0$. One has for $u \in C_0^\infty(\mathbb{R}_+^n)$:

$$\begin{aligned} \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \left(\frac{5}{2} - \frac{5}{2} \varepsilon \right) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + \left(\frac{81}{16} - \frac{9}{4} \frac{5}{2} \varepsilon \right) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ \geq C \left(\int_{\mathbb{R}_+^n} |u|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}}. \end{aligned}$$

In particular, when $\varepsilon = \frac{9}{10}$, we obtain

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx \geq C \left(\int_{\mathbb{R}_+^n} |u|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}}$$

which improves (8) since (see, for instance, [31]):

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx.$$

2 Proof of Theorem 1 and Theorem 3

Proof (Proof for Theorem 1) For $u \in C_0^\infty(\mathbb{R}_+^n)$, we define

$$Su = -\Delta u + \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} - \beta \frac{u}{x_n^2}$$

Then its formal adjoint is

$$S^*v = -\Delta v - \frac{\alpha}{x_n} \frac{\partial v}{\partial x_n} - (\beta - \alpha) \frac{v}{x_n^2}$$

and hence

$$\begin{aligned} S^*Su &= \Delta \left[\Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] + \frac{\alpha}{x_n} \frac{\partial}{\partial x_n} \left[\Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] \\ &\quad + \frac{(\beta - \alpha)}{x_n^2} \left[\Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right]. \end{aligned}$$

As a consequence,

$$\begin{aligned} &\int_{\mathbb{R}_+^n} (S^*Su) u dx \\ &= \int_{\mathbb{R}_+^n} \left[\Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] \Delta u dx + \int_{\mathbb{R}_+^n} \frac{\alpha}{x_n} \frac{\partial}{\partial x_n} \left[\Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] u dx \\ &\quad + (\beta - \alpha) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left[\Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] u dx \\ &= \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \alpha \int_{\mathbb{R}_+^n} \frac{1}{x_n} \frac{\partial u}{\partial x_n} \Delta u dx + \beta \int_{\mathbb{R}_+^n} \frac{u}{x_n^2} \Delta u dx \\ &\quad - \alpha \int_{\mathbb{R}_+^n} \left[\Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] \left(\frac{1}{x_n} \frac{\partial u}{\partial x_n} - \frac{u}{x_n^2} \right) dx \\ &\quad + (\beta - \alpha) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left[\Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] u dx \\ &= \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - 2\alpha \int_{\mathbb{R}_+^n} \frac{1}{x_n} \frac{\partial u}{\partial x_n} \Delta u dx + (\beta + \alpha + \beta - \alpha) \int_{\mathbb{R}_+^n} \frac{u}{x_n^2} \Delta u dx \\ &\quad + \alpha^2 \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx - \left(\alpha^2 + \alpha\beta + \alpha(\beta - \alpha) \right) \end{aligned}$$

$$\begin{aligned}
& \int_{\mathbb{R}_+^n} \frac{1}{x_n^3} \frac{\partial u}{\partial x_n} u dx + (\alpha\beta + \beta(\beta - \alpha)) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\
&= \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - 2\alpha \int_{\mathbb{R}_+^n} \frac{1}{x_n} \frac{\partial u}{\partial x_n} \Delta u dx + 2\beta \int_{\mathbb{R}_+^n} \frac{u}{x_n^2} \Delta u dx \\
&+ \alpha^2 \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx - 2\alpha\beta \int_{\mathbb{R}_+^n} \frac{1}{x_n^3} \frac{\partial u}{\partial x_n} u dx + \beta^2 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx
\end{aligned}$$

Noting that by integrations by parts, we get

$$\begin{aligned}
\int_{\mathbb{R}_+^n} \frac{1}{x_n} \frac{\partial u}{\partial x_n} \Delta u dx &= \int_{\mathbb{R}_+^n} \left(\frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 - \frac{1}{2} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} \right) dx, \\
\int_{\mathbb{R}_+^n} \frac{u}{x_n^2} \Delta u dx &= - \int_{\mathbb{R}_+^n} \nabla u \nabla \left(\frac{u}{x_n^2} \right) dx \\
&= - \int_{\mathbb{R}_+^n} \nabla u \left[\frac{1}{x_n^2} \nabla u - \frac{2}{x_n^3} u e_n \right] dx \\
&= - \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + 2 \int_{\mathbb{R}_+^n} \frac{1}{x_n^3} u \frac{\partial u}{\partial x_n} dx \\
&= - \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + 3 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}_+^n} \frac{1}{x_n^3} u \frac{\partial u}{\partial x_n} dx &= - \int_{\mathbb{R}_+^n} \frac{1}{x_n^3} u \frac{\partial u}{\partial x_n} dx + 3 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\
&= \frac{3}{2} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx.
\end{aligned}$$

Hence

$$\begin{aligned}
& \int_{\mathbb{R}_+^n} (S^* Su) u dx \\
&= \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - 2\alpha \left[\int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx - \frac{1}{2} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx \right] \\
&\quad + 2\beta \left[- \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + 3 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \right] \\
&\quad + \alpha^2 \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx - 3\alpha\beta \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx + \beta^2 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\
&= \int_{\mathbb{R}_+^n} |\Delta u|^2 dx + (\alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + (\alpha^2 - 2\alpha) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx \\
&\quad + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx.
\end{aligned}$$

Therefore, since

$$\int_{\mathbb{R}_+^n} (S^* Su) u dx = \int_{\mathbb{R}_+^n} \|Su\|^2 dx,$$

we obtain

$$\begin{aligned}
& \int_{\mathbb{R}_+^n} |\Delta u|^2 dx + (\alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\
&\quad + (\alpha^2 - 2\alpha) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx \\
&= \int_{\mathbb{R}_+^n} \left| \Delta u - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right|^2 dx.
\end{aligned}$$

Now, let us denote

$$Tu = -\frac{\partial^2 u}{\partial x_n^2} + \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} - \beta \frac{u}{x_n^2}$$

then

$$T^*v = -\frac{\partial^2 v}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial v}{\partial x_n} - (\beta - \alpha) \frac{v}{x_n^2}$$

and

$$\begin{aligned} & \int_{\mathbb{R}_+^n} (T^*Tu) u dx \\ &= \int_{\mathbb{R}_+^n} \frac{\partial^2}{\partial x_n^2} \left[\frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] u dx \\ &+ \alpha \int_{\mathbb{R}_+^n} \frac{1}{x_n} \frac{\partial}{\partial x_n} \left[\frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] u dx \\ &+ (\beta - \alpha) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left[\frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] u dx \\ &= \int_{\mathbb{R}_+^n} \left[\frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] \frac{\partial^2 u}{\partial x_n^2} dx \\ &- \alpha \int_{\mathbb{R}_+^n} \left[\frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] \left(\frac{1}{x_n} \frac{\partial u}{\partial x_n} - \frac{u}{x_n^2} \right) dx \\ &+ (\beta - \alpha) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left[\frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right] u dx \\ &= \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx - 2\alpha \int_{\mathbb{R}_+^n} \frac{1}{x_n} \frac{\partial u}{\partial x_n} \frac{\partial^2 u}{\partial x_n^2} dx + 2\beta \int_{\mathbb{R}_+^n} \frac{u}{x_n^2} \frac{\partial^2 u}{\partial x_n^2} dx \\ &+ \alpha^2 \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx - 2\alpha\beta \int_{\mathbb{R}_+^n} \frac{1}{x_n^3} \frac{\partial u}{\partial x_n} u dx + \beta^2 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \end{aligned}$$

By integrations by parts, we get

$$\int_{\mathbb{R}_+^n} \frac{1}{x_n} \frac{\partial u}{\partial x_n} \frac{\partial^2 u}{\partial x_n^2} dx = \frac{1}{2} \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx$$

and

$$\begin{aligned} \int_{\mathbb{R}_+^n} \frac{u}{x_n^2} \frac{\partial^2 u}{\partial x_n^2} dx &= - \int_{\mathbb{R}_+^n} \frac{\partial u}{\partial x_n} \frac{\partial}{\partial x_n} \left(\frac{u}{x_n^2} \right) dx \\ &= - \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + 2 \int_{\mathbb{R}_+^n} \frac{1}{x_n^3} u \frac{\partial u}{\partial x_n} dx \\ &= - \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + 3 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\mathbb{R}_+^n} (T^* T u) u dx \\ &= \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx - \alpha \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + 2\beta \left[- \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + 3 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \right] \\ &+ \alpha^2 \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx - 2\alpha\beta \frac{3}{2} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx + \beta^2 \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ &= \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx + (\alpha^2 - \alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx. \end{aligned}$$

Noting that

$$\int_{\mathbb{R}_+^n} (T^* T u) u dx = \int_{\mathbb{R}_+^n} \|Tu\|^2 dx,$$

we deduce

$$\begin{aligned} &\int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} \right|^2 dx + (\alpha^2 - \alpha - 2\beta) \int_{\mathbb{R}_+^n} \frac{1}{x_n^2} \left| \frac{\partial u}{\partial x_n} \right|^2 dx + (6\beta - 3\alpha\beta + \beta^2) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\ &= \int_{\mathbb{R}_+^n} \left| \frac{\partial^2 u}{\partial x_n^2} - \frac{\alpha}{x_n} \frac{\partial u}{\partial x_n} + \beta \frac{u}{x_n^2} \right|^2 dx. \end{aligned}$$

□

We now provide a proof for Theorem 3

Proof (Proof for Theorem 3) For $\varepsilon \leq 1$, we have

$$\begin{aligned}
 & \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \left(\frac{5}{2} - \frac{5}{2}\varepsilon \right) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + \left(\frac{81}{16} - \frac{9}{4} \frac{5}{2}\varepsilon \right) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\
 &= \varepsilon \left(\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \right) \\
 &+ (1 - \varepsilon) \left[\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{5}{2} \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + \frac{81}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \right] \\
 &\geq \varepsilon \left(\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \right) \\
 &\geq C \left(\int_{\mathbb{R}_+^n} |u|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}}.
 \end{aligned}$$

Now, for $\varepsilon > 1$, we write

$$\begin{aligned}
 & \int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \left(\frac{5}{2} - \frac{5}{2}\varepsilon \right) \int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx + \left(\frac{81}{16} - \frac{9}{4} \frac{5}{2}\varepsilon \right) \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \\
 &= \left(\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \right) + \left(\frac{5}{2}\varepsilon - \frac{5}{2} \right) \left[\int_{\mathbb{R}_+^n} \frac{|\nabla u|^2}{x_n^2} dx - \frac{9}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \right] \\
 &\geq \left(\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \right) \\
 &\geq C \left(\int_{\mathbb{R}_+^n} |u|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}}.
 \end{aligned}$$

Here we used the Rellich–Sobolev–Maz’ya type inequality (see [3, Chapter 6]):

$$\int_{\mathbb{R}_+^n} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^4} dx \geq C \left(\int_{\mathbb{R}_+^n} |u|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}}.$$

□

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